18.06 MIDTERM 2

(Note: those of you starting at 11+x hours, please add x hours to all the times below)

- *10:55 - exam available on Canvas (email jhahn01@mit.edu ASAP if you can't access it)*
- *11:05 - you may start reading the problems and working on them*
- *11:55 - you must stop writing at this time*
- *12:10 - Gradescope upload cut-off. If you can't upload your finished exam to Gradescope for technical reasons, please email it to jhahn01@mit.edu before this time*

NO collaboration or written/electronic/online sources allowed, except for our course materials (lecture and recitation notes, problem sets and review session problems + solutions).

DOWNLOAD or **DOWNLOAD + PRINT** this exam before the official start time. You can either annotate the PDF file, or physically write on paper (if you need extra sheets of paper, please write your name and the problem number on **ALL** pages that you want graded).

UPLOAD or **SCAN + UPLOAD** your exam to Gradescope after time is up and pencils are down. Make sure that the pages are easily legible (e.g. good camera quality and angle).

You **MUST** show your work to receive credit. **JUSTIFY EVERYTHING**. Just giving a correct answer without an explanation of what led you to it will be **SEVERELY** penalized.

The three problems have 4,4 and 3 parts, respectively.

NAME: ______________________________

MIT ID NUMBER: ______________________________

RECITATION INSTRUCTOR: ______________________________

PROBLEM 1 **NAME:** ____________________

(1) Use the Gram-Schmidt process to convert the vectors:

$$\begin{bmatrix} 4 \\ 6 \\ 12 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ -9 \\ -4 \end{bmatrix}$$

into an orthonormal basis of the vector space they span. **Show all your steps!** *(15 pts)*

NAME: ____________________

(2) Use the previous part to obtain a factorization:

$$\boxed{A = QR} \qquad \text{of the matrix} \qquad A = \begin{bmatrix} 4 & 1 \\ 6 & -9 \\ 12 & -4 \end{bmatrix}$$

where Q has orthonormal columns and R is an upper triangular square matrix. **Show all the steps of your argument, and explain how it derives from part (1)!** *(10 pts)*

NAME: ____________________

(3) With the matrix Q as on the previous page, consider the linear transformation:

$$f : \mathbb{R}^2 \to \mathbb{R}^3, \qquad f(\mathbf{v}) = Q\mathbf{v}$$

Suppose you have any two orthogonal (i.e. perpendicular) vectors $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2$. Use linear algebra to <u>prove</u> that the vectors $f(\mathbf{v}_1), f(\mathbf{v}_2) \in \mathbb{R}^3$ are also orthogonal. *(5 pts)*

(4) <u>Guess</u> (no explanation needed, just this once) an eigenvector $\mathbf{a} \neq 0$ of the matrix R from the previous page, and the corresponding eigenvalue. Draw the linear transformation:

$$g : \mathbb{R}^2 \to \mathbb{R}^2, \qquad g(\mathbf{w}) = R\mathbf{w}$$

on a picture of $\mathbb{R}^2$, by drawing the eigenvector $\mathbf{a}$ and showing where the function g sends both $\mathbf{a}$ and any other vector in $\mathbb{R}^2$ of your choice, linearly independent from $\mathbf{a}$. *(5 pts)*

PROBLEM 2 **NAME:** ____________________

(1) Use row operations (i.e. $\pm$ product of pivots) to compute the determinant of the matrix:

$$\begin{bmatrix} 2 & 3 & 1 \\ -4 & -6 & -1 \\ 1 & 2 & 1 \end{bmatrix}$$

(if instead of row operations, you use any other method to compute the determinant, you will lose between 50% and 70% of the points). *(10 pts)*

NAME: ____________________

(2) Compute $\boxed{z}$ from the system of equations below using Cramer's rule:

$$\begin{bmatrix} 2 & 0 & 1 \\ -3 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

You must explicitly write z as a ratio of determinants, and compute these determinants via cofactor expansion. *(10 pts)*

NAME: ____________________

(3) Consider an arbitrary $n \times n$ matrix A. We have the following formula for the inverse:

$$\boxed{A^{-1} = \frac{1}{\det A} \cdot C^T}$$

where the (i, j) entry of the cofactor matrix C is $(-1)^{i+j}$ times the determinant of the $(n-1) \times (n-1)$ matrix obtained by removing row i and column j from A.

Find and prove a formula for $\det C$ in terms of $\det A$. **Explain all your steps!** *(5 pts)*

NAME: ____________________

(4) The big formula for the determinant of:

$$\begin{bmatrix} 0 & 0 & 0 & a_{14} & a_{15} & a_{16} \\ 0 & 0 & 0 & 0 & a_{25} & a_{26} \\ 0 & 0 & 0 & 0 & 0 & a_{36} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ 0 & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} \\ 0 & 0 & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix}$$

is a sum of $6! = 720$ terms. Explain why 719 of these terms are zero, but one is non-zero (assuming the a_{ij}'s are non-zero themselves). What is this non-zero term? *(10 pts)*

PROBLEM 3 **NAME:** ____________________

Consider the linear transformation $\phi : \mathbb{R}^3 \to \mathbb{R}^3$, $\phi(\mathbf{v}) = A\mathbf{v}$, where:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & -1 \\ 1 & 1 & 2 \end{bmatrix}$$

We'll give you the following bits of information:

- ϕ fixes a certain (two-dimensional) plane $P \subset \mathbb{R}^3$, i.e. $\phi(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v} \in P$
- ϕ rescales a certain line $\ell \subset \mathbb{R}^3$ by a factor of 2, i.e. $\phi(\mathbf{w}) = 2\mathbf{w}$ for all $\mathbf{w} \in \ell$

(1) What are the eigenvalues of A, and their algebraic/geometric multiplicities (explain how you know, based on the information given in the bullets above)? *(10 pts)*

NAME: ____________________

(2) Compute a basis for the line ℓ in the previous part. *(5 pts)*

NAME: ____________________

(3) Find an invertible matrix V and a diagonal matrix D such that:

$$\begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix} = VDV^{-1}$$

(in other words, diagonalize the 2×2 matrix in the left hand side). *(15 pts)*

Hints:

- *It's enough to compute one eigenvalue and its eigenvector (**show all your steps**). Then you can invoke a general principle (**say what it is**) to get the other eigenvalue/eigenvector*
- *Don't be afraid if the answer will involve complex numbers*
- *Don't forget to tell us what V and D are*